

# Final

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## Topology

1. Recall that a function  $f$  from a topological space  $X$  into a topological space  $Y$  is called continuous if the inverse image of an open subset of  $Y$  is an open subset of  $X$ . Is the function from  $\mathbb{R}$  to  $\mathbb{R}$  defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if not} \end{cases}$$

continuous? (The topology on  $\mathbb{R}$  is the usual topology. I want a proof and not only the answer).

2. Let  $X$  be an arbitrary set. Call a subset  $U$  of  $X$  open if either  $U = \emptyset$  or  $X \setminus U$  is finite. Show that this defines a topology on  $X$ . What is the closure of an arbitrary subset  $Y$  of  $X$ . (Recall that the closure of a subset is the smallest closed subset containing the given subset).

3. Recall that a subset  $K$  of a topological space  $X$  is called compact if for any family  $(U_i)_i$  of open subsets of  $X$  whenever  $K \subseteq \cup_i U_i$ , then there are  $U_{i_1}, \dots, U_{i_n}$  such that  $K \subseteq U_{i_1} \cup \dots \cup U_{i_n}$ .

Show that if  $X$  is a compact topological space then any closed subset of  $X$  is also compact.

## Set Theory

4. Let  $G$  be a group. Let  $g \in G$ . Show that there is a maximal subgroup of  $G$  not containing  $g$ .

5. Let  $X$  be a set. Show that there is no bijection between  $X$  and  $\mathcal{P}(X)$ .

6. Show that the union of two countable sets is countable.

## Model Theory

7. Find all the automorphisms of  $\langle \mathbb{R}, +, \times \rangle$ .

8. Find a formula in one variable in the language  $\{+, \times\}$  such that the elements of  $\mathbb{R}[X]$  that satisfy this formula are exactly the polynomials of degree 1.

## Cauchy Sequences

9. Show that a Cauchy sequence in  $\mathbb{Q}$  is bounded.

10. Let  $(x_n)_n$  be a Cauchy sequence in  $\mathbb{Q}$  that does not converge to 0. Show that  $(1/x_n)_n$  is a Cauchy sequence. Show that if  $\lim_{n \rightarrow \infty} x_n = a \neq 0$  then  $\lim_{n \rightarrow \infty} 1/x_n = 1/a$ .