Group Theory Problems
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1 Matrix Groups

1.1 Metric Structure on $\mathrm{Mat}_{n\times n}(\mathbb{C})$

Problem 1.1 *Let A be an $n \times n$ complex matrix. Show that $|v| = |vA|$ for all vectors v if and only if $A^{-1} = \overline{A}^t$. Here, $|v| = (|v_1|^2 + \ldots + |v_n|^2)^{1/2}$. The set of such matrices form a group.*

Proof: Since $|v|^2 = v\overline{v}^t$, we have $v\overline{v}^t = |v|^2 = |vA|^2 = vA\overline{(vA)}^t = vA\overline{A}^t\overline{v}^t$. Taking special values for v, we easily get $A\overline{A}^t = \mathrm{Id}$. □

Problem 1.2 *Show that the matrices*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

are conjugate in $\mathrm{SL}_2(\mathbb{C})$ but not in $\mathrm{SL}_2(\mathbb{R})$.

Proof: The matrix

$$X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

should do the job. If $Y \in \mathrm{SL}_2(\mathbb{R})$ does the same job, then XY^{-1} in the centralizer of one of the elements. This must be enough to get a contradiction. □

Problem 1.3 *Show that* $\mathrm{PGL}_2(\mathbb{C})$ *and* $\mathrm{PSL}_2(\mathbb{C})$ *are isomorphic. Are* $\mathrm{PGL}_2(\mathbb{R})$ *and* $\mathrm{PSL}_2(\mathbb{R})$ *isomorphic?*

Problem 1.4 *Show that* tr *has the following properties:*
i. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
ii. $\mathrm{tr}(BAB^{-1}) = \mathrm{tr}(A)$.
iii. $\mathrm{tr}(\lambda A) = \lambda\,\mathrm{tr}(A)$.
iv. $\mathrm{tr}(A^t) = \mathrm{tr}(A)$.

Given any matrix A, define $A^\star = \overline{A}^t$ (**Hermitian Transpose**) and $[A,B] = \mathrm{tr}(AB^\star)$.

Problem 1.5 *Show that*
i. $[A,A] \geq 0$ *and* $[A,A] = 0$ *if and only if* $A = 0$.
ii. $[\ ,\ ]$ *is bilinear in the first component.*
iii. $[B,A] = \overline{[A,B]}$.

Proof: All follows from the fact that if $A = (a_{ij})_{i,j}$ and $B = (b_{ij})_{i,j}$, then $[A,B] = \sum_{i,j} a_{ij}\overline{b}_{ij}$. □

Thus $[\ ,\ ]$ is a **scalar product** on the vector space $\mathrm{Mat}_{n\times n}(\mathbb{C})$.
Given a scalar product $[\ ,\ ]$ one can define $\|A\| = [A,A]^{1/2}$.

Problem 1.6 *Show the following*
iv. $\|A\| \geq 0$ *and* $\|A\| = 0$ *if and only if* $A = 0$.
v. $\|\lambda A\| = |\lambda|\|A\|$.
vi. $|[A,B]| \leq \|A\|\,\|B\|$.
vii. $\|A+B\| \leq \|A\| + \|B\|$.

Proof: iv and v are easy. For vi, let $C = \lambda A - \mu B$ where $\lambda = [B,A]$ and $\mu = \|A\|^2$. Use $\|C\|^2 \geq 0$. For (vii) use this and the fact that

$$\|A+B\|^2 = \|A\|^2 + [A,B] + [B,A] + \|B\|^2.$$

□

The properties iv, v and vii say that $\|\ \|$ is a **norm** on the vector space $\mathrm{Mat}_{n\times n}(\mathbb{C})$.

Problem 1.7 *Show that any norm* $\|\ \|$ *induces a metric by defining* $d(A,B) = \|A-B\|^{1/2}$.

Problem 1.8 *Show that if* $n = 2$ *then we have*
viii. $|\det(A)|\,\|A^{-1}\| = \|A\|$.
ix. $\|AB\| \leq \|A\|\cdot\|B\|$.
x. $2|\det(A)| \leq \|A\|^2$.
Do any of these hold if $n \neq 2$*?*

Problem 1.9 *Show that the convergence in* $\mathrm{GL}_n(\mathbb{C})$ *with respect to this norm is the pointwise convergence.*

Problem 1.10 *Show that* $\mathrm{GL}_n(\mathbb{C})$ *is a topological group.*

Problem 1.11 *Show that for* A, $B \in \mathrm{SL}_2(\mathbb{C})$

i. $\mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$,

ii. $tr(BAB) + \mathrm{tr}(A) = \mathrm{tr}(B)\,\mathrm{tr}(AB)$,

iii. $\mathrm{tr}^2(A) + \mathrm{tr}^2(B) + \mathrm{tr}^2(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B)\,\mathrm{tr}(AB) + 2 + \mathrm{tr}(ABA^{-1}B^{-1})$.

iv. Replacing B *by* $A^n B$ *in (i), obtain a formula for* $\mathrm{tr}(A^n B)$ *as a function of* $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\mathrm{tr}(AB)$ *and* n.

Problem 1.12 *Show the following:*

i. $\mathrm{GL}_n(\mathbb{C})$ *is an open but not a closed subset of* $\mathrm{Mat}_{n\times n}(\mathbb{C})$.

ii. $\mathrm{SL}_n(\mathbb{C})$ *is a closed but not an open nor a compact subset of* $\mathrm{Mat}_{n\times n}(\mathbb{C})$.

iii. $\mathrm{GL}_n(\mathbb{R})$ *is disconnected.*

iv. $\mathrm{GL}_n(\mathbb{C})$ *is connected. (Hint: Use Jordan Canonical Form and show that every element is path connected to* Id_n*).*

v. $\{A : \mathrm{tr}(A) = 1\}$ *is closed but not compact.*

1.2 Discrete Subgroups of $\mathrm{GL}_n(\mathbb{C})$

We say that a subgroup G of $\mathrm{GL}_n(\mathbb{C})$ is **discrete** if and only if the induced topology on G is discrete, i.e. if for any $g \in G$ there is an open subset U of $\mathrm{GL}_n(\mathbb{C})$ such that $U \cap G = \{g\}$.

Problem 1.13 *Let* $G \leq \mathrm{GL}_n(\mathbb{C})$. *Show that the following are equivalent:*

i. G *is discrete.*

ii. $\inf\{\|g - 1\| : g \in G^*\} > 0$.

iii. All Cauchy sequences in G *are eventually constant sequences.*

iv. Any sequence that converges to 1 is eventually 1.

v. G *has no limit points.*

Problem 1.14 *Let* $G \leq SL_n(\mathbb{C})$.

i. Show that G *is discrete if and only if for each* $k \in \mathbb{N}$, *the set* $\{g \in G : \|g\| \leq k\}$ *is finite.*

ii. Conclude that a discrete subgroup of $\mathrm{SL}_n(\mathbb{C})$ *is countable.*

iii. Show that (i) is false for $\mathrm{GL}_n(\mathbb{C})$, *but (ii) holds.*

Problem 1.15 *i. Show that any subgroup of a discrete group is discrete.*

ii. Show that a conjugate of a discrete subgroup is discrete.

Problem 1.16 *Show that* $\mathrm{SL}_n(\mathbb{Z})$ *and* $\mathrm{SL}_n(\mathbb{Z}[i])$ *are discrete subgroups of* $\mathrm{SL}_n(\mathbb{C})$.

Problem 1.17 *Find all discrete subgroups of* $\mathrm{GL}_2(\mathbb{C})$ *that contain only diagonal elements.*

Solution. First note that a subgroup of a discrete group is discrete. Since $\mathrm{GL}_2(\mathbb{C}) \simeq \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^*$, we have to find the discrete subgroups of $\mathbb{C}^*$. Since $\mathbb{C}^* \simeq \mathbb{R}^{>1} \times S^1$, it is enough to find the discrete subgroups of $\mathbb{R}^{>1}$ and S^1. Discrete subgroups of S^1 are finite (hence cyclic) because S^1 is compact. Let us find the discrete subgroups of $\mathbb{R}^{>1}$. It is clear that one-generated subgroups of $\mathbb{R}^{>0}$ are discrete. It is also clear that a finite extension of a discrete group is discrete.

Claim. [Seyfi] *Let $r, s \in \mathbb{R}^{>0}$. Then $\langle r, s\rangle$ is discrete if and only if $r^n = s^m$ for some positive integers n and m.*

($\Leftarrow$) If $r^n = s^m$, then $\langle r, s\rangle$ is a finite extension of $\langle r^n\rangle$ and we are done.

($\Rightarrow$) Assume $r^n \neq s^m$ for any positive integers n and m. We will show that 1 is the accumulation point of the group $\langle r, s\rangle$, proving that the group $\langle r, s\rangle$ is not discrete. We may assume that r and s are greater than 1 and that they are distinct. For any $n \in \mathbb{N}$, look at the set $\{m \in \mathbb{N} : r^m/s^n < r/s\}$. Since $r > 1$, this set has a greatest element, say m. Then $r^m/s^n < r/s \leq r^{m+1}/s^n < r^2/s$. Thus there are infinitely many pairs of integers (m, n) such that $r/s \leq r^m/s^n \leq r^2/s$. These values r^m/s^n are necessarily distinct. Thus the set $\{r^m/s^n : r/s \leq r^m/s^n \leq r^2/s\}$ has an accumulation point α. If $\alpha = 1$ we are done. Otherwise, choose a sequence $\alpha_k \in \langle r, s\rangle$ of distinct elements that converge to α. Then the sequence $\alpha_k\alpha_{k+1}^{-1}$ of elements $\neq 1$ converges to 1. □

It follows that a finitely generated discrete subgroup of $\mathbb{R}^{>0}$ is a finite extension of a one-generated group.

Problem 1.18 *Let $G \leq \mathrm{GL}_2(\mathbb{C})$ contain a discrete subgroup of finite index. Show that G is discrete.*

References

[**B**] Alan F. Beardon, **The Geometry of Discrete Groups**, Springer 1995.

2 Geometric Automorphism Groups

Problem 2.1 (Isometries of $\mathbb{R}$) *Let $G := \{g : \mathbb{R} \longrightarrow \mathbb{R} : |x - y| = |g(x) - g(y)|\}$. We will show that G is a group under composition and we will find all the elements of G. Note that G is clearly closed under composition and $Id_{\mathbb{R}} \in G$. Also if $g \in G$ is a bijection, then $g^{-1} \in G$ as well. To show that G is a group, we just need to show that the elements of G are bijections.*

i. *Show that any element of G is one-to-one.*

ii. *For $a \in \mathbb{R}$, let $\tau_a : \mathbb{R} \longrightarrow \mathbb{R}$ be given by $\tau_a(x) = x + a$ (translation). Show that $\tau_a \in G$. Show that the set $T = \{\tau_a : a \in \mathbb{R}\}$ is a group under composition.*

iii. *For $\epsilon = \pm 1$, let $\rho_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$ be given by $\rho_\epsilon(x) = \epsilon x$. Show that $\rho_\epsilon \in G$. Show that the set $R = \{\rho_1, \rho_{-1}\}$ is a group under composition.*

iv. *Show that for any $g \in G$ there are unique $\epsilon \in \{1, -1\}$ and $a \in \mathbb{R}$ such that $g = \tau_a \circ \rho_\epsilon$. Conclude that the elements of G are bijections.*

v. *Conclude that G is a group under composition.*

The group G is called the **group of isometries** *of $\mathbb{R}$.*

Problem 2.2 (Isometries of $\mathbb{R}^2$) *Let $G := \{g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : |x-y| = |g(x) - g(y)|\}$. We will show that G is a group under composition and we will find all the elements of G explicitly. Note that G is clearly closed under composition and $Id_{\mathbb{R}} \in G$. Also if $g \in G$ is a bijection, then $g^{-1} \in G$ as well. To show that G is a group, we just need to show that the elements of G are bijections.*

Note that G is the set of functions from $\mathbb{R}^2$ into $\mathbb{R}^2$ that send any circle into a circle of the same radius.

i. *Show that any element of G is one-to-one.*

ii. *For $(a,b) \in \mathbb{R}^2$, let $\tau_{(a,b)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $\tau_{(a,b)}(x,y) = (x+a, y+b)$ (translation). Show that $\tau_{(a,b)} \in G$. Show that the set $T := \{\tau_{(a,b)} : (a,b) \in \mathbb{R}\}$ is a group under composition.*

iii. *For $\theta \in [0, 2\pi)$, let $\rho_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the clockwise rotation of angle θ around the center $(0,0)$, i.e. $\rho_\theta(x,y) = (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta))$. Show that $\rho_\theta \in G$. Show that the set $R := \{\rho_\theta : \theta \in [0, 2\pi)\}$ is a group under composition.*

iv. *For $\epsilon = \pm 1$, let $\sigma_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$ be given by $\sigma_\epsilon(x,y) = (x, \epsilon y)$. Show that $\sigma_\epsilon \in G$. Show that the set $S = \{\sigma_1, \sigma_{-1}\}$ is a group under composition.*

v. *Show that for any $g \in G$ there is a unique $(a,b) \in \mathbb{R}^2$ such that such that $(\tau_{a,b} \circ g)(0,0) = (0,0)$.*

vi. *Show that for any $g \in G$ such that $g(0,0) = (0,0)$ there is there is a unique $\theta \in [0, 2\pi)$ such that $(\rho_\theta \circ g)(0,0) = (0,0)$ and $(\rho_\theta \circ g)(1,0) = (1,0)$*

vii. *Show that for any $g \in G$ such that $g(0,0) = (0,0)$ and $g(1,0) = (1,0)$ we have $g(0,1) = (0,\epsilon)$ for some $\epsilon = \pm 1$. Conclude that for such g there is a unique $\epsilon = \pm 1$ such that $(s_\epsilon \circ g)(0,0) = (0,0)$, $(s_\epsilon \circ g)(1,0) = (1,0)$ and $(s_\epsilon \circ g)(0,1) = (0,1)$.*

vii. *Show that if $g \in G$ is such that $g(0,0) = (0,0)$, $g(1,0) = (1,0)$ and $g(1,0) = (1,0)$, then $g = \mathrm{Id}_{\mathbb{R}^2}$.*

viii. *Conclude that for any $g \in G$ there are unique $(a,b) \in \mathbb{R}^2$, $\theta \in [0, 2\pi)$ and $\epsilon = \pm 1$ such that $g = \tau_{(a,b)} \circ \rho_\theta \circ \sigma_\epsilon$. Conclude that g is a bijection.*

ix. *Conclude that G is a group under composition.*

The group G is called the **group of isometries** *of $\mathbb{R}$.*

x. *Conclude that an element of G sends a line onto a line.*

Problem 2.3 (Affine Transformations of $\mathbb{R}^2$) *Let $G := \{g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : g \text{ is a bijection and sends a line into a line}\}$. It is clear that G is a group under composition. We will find all the elements of G explicitly. As we have noticed in the end of Problem 2.2, the elements of the group defined there are in G. Thus the groups T and R are subsets of G. We adopt the same terminology.*

i. *Show that an element of G respect the parallelism, i.e. sends two parallel lines onto two parallel lines.*

ii. *For $c \in \mathbb{R}^{>0}$, let $h_c : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $h_c(x,y) = (cx, y)$. Show that $h_c \in G$ and that the set $H := \{h_c : c \in \mathbb{R}^{>0}\}$ is a group under composition.*

iii. *For* $d \in \mathbb{R}$ *define* $u_d : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *by* $u_d(x, y) = (x + dy, y)$. *Show that* $u_d \in G$ *and that the set* $U := \{t_d : d \in \mathbb{R}\}$ *is a group under composition.*

iv. *For a bijection phi* : $\mathbb{R} \longrightarrow \mathbb{R}$ *satisfying the properties* $\phi(x + y) = \phi(x) + \phi(y)$, $\phi(xy) = \phi(x)\phi(y)$ *and* $\phi(1) = 1$, *define the map* $\alpha_\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *by* $\alpha_\phi(x, y) = (\phi(x), \phi(y))$. *Show that* $\alpha_\phi \in G$ *and that the set* $A := \{\alpha_\phi : \phi \text{ as above}\}$ *is a group under composition.*

v. *Show that for any* $g \in G$ *there is a unique* $(a, b) \in \mathbb{R}^2$ *such that such that* $(\tau_{a,b} \circ g)(0, 0) = (0, 0)$.

vi. *Show that for any* $g \in G$ *such that* $g(0, 0) = (0, 0)$ *there is there is a unique* $\theta \in [0, 2\pi)$ *such that* $(\rho_\theta \circ g)(0, 0) = (0, 0)$ *and* $(\rho_\theta \circ g)(1, 0)$ *is on the positive side of the* x*-axis.*

vii. *Show that for any* $g \in G$ *such that* $g(0, 0) = (0, 0)$ *and* $g(1, 0)$ *is on the positive side of the* x*-axis, there is a unique* $c \in \mathbb{R}^{>0}$ *such that* $(h_c \circ g)(0, 0) = (0, 0)$ *and* $(h_c \circ g)(1, 0) = (1, 0)$.

viii. *Show that for any* $g \in G$ *such that* $g(0, 0) = (0, 0)$ *and* $g(1, 0) = (1, 0)$ *there is a unique* $d \in \mathbb{R}$ *such that* $(u_d \circ g)(0, 0) = (0, 0)$, $(u_d \circ g)(1, 0) = (1, 0)$ *and* $(u_d \circ g)(0, 1) = (0, 1)$.

ix. *Show that if* $g \in G$ *satisfies* $g(0, 0) = (0, 0)$, $g(1, 0) = (1, 0)$ *and* $g(0, 1) = (0, 1)$, *then* $g \in A$. *(Hint: Express the addition and multiplication of real numbers geometrically and use part (i)).*

x. *Show that for any* $g \in G$ *there are unique* $(a, b) \in \mathbb{R}^2$, $\theta \in [0, 2\pi)$, $c \in \mathbb{R}^{>0}$, $d \in \mathbb{R}$ *and bijection phi* : $\mathbb{R} \longrightarrow \mathbb{R}$ *satisfying the properties* $\phi(x + y) = \phi(x) + \phi(y)$, $\phi(xy) = \phi(x)\phi(y)$ *and* $\phi(1) = 1$, *such that* $g = \tau_{(a,b)} \circ \rho_\theta \circ h_c \circ u_d \circ \alpha_\phi$.

The group G *is called the* **affine transformations** *of* $\mathbb{R}^2$.

Problem 2.4 *On* $\mathbb{R}^2$ *define the metric* $d((x, y), (z, t)) := |z - x| + |t - y|$. *Let* G *be the group of isometries of this metric. Find the elements of* G.

Problem 2.5 *On* $\mathbb{R}^2$ *define the metric* $d((x, y), (z, t)) := \max(|z - x|, |t - y|)$. *Let* G *be the group of isometries of this metric. Find the elements of* G.