Analysis (Math 162)
Final

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Do not use symbols such as $\Rightarrow$, $\forall$.
Make full sentences.
Write legibly. Use correct punctuation.
Explain your ideas.

I. Convergent Sequences. For each of the topological spaces (X, τ), describe the convergent sequences and discuss the uniqueness of their limits.

1. $\tau = \wp(X)$. ($\wp(X)$ is the set of all subsets of X, 2 pts.).

   Answer: Only the eventually constant sequences converging to that constant.

2. $\tau = \{\emptyset, X\}$ (2 pts.).

   Answer: All sequences converge to all elements.

3. $a \in X$ is a fixed element and τ is the set of all subsets of X that do not contain a, together with X of course. (5 pts.)

   Answer: First of all, all sequences converge to a. Second: If a sequence converges to $b \neq a$, then the sequence must be eventually the constant b.

4. $a \in X$ is a fixed element and τ is the set of all subsets of X that contain a, together with $\emptyset$ of course. (5 pts.)

   Answer: Only the eventually constant sequences converge to a. A sequence converge to $b \neq a$ if and only if the sequence eventually takes only the two values a and b.

5. τ is the set of all cofinite subsets of X, together with the $\emptyset$ of course. (6 pts.)

   Answer: All the sequences without infinitely repeating terms converge to all elements. Eventually constant sequences converge to the constant. There are no others.

II. Subgroup Topology on $\mathbb{Z}$. Let $\tau = \{n\mathbb{Z}+m : n,\, m \in \mathbb{Z},\, n \neq 0\} \cup \{\emptyset\}$. We know that $(\mathbb{Z}, \tau)$ is a topological space.

1. Let $a \in \mathbb{Z}$. Is $\mathbb{Z} \setminus \{a\}$ open in τ? (5 pts.)

   Answer: Yes. $\cup_{n\neq 0,\pm 1} n\mathbb{Z} \cup (3\mathbb{Z}+2) = \mathbb{Z} \setminus \{1\}$.Translating this set by $a-1$, we see that $\mathbb{Z} \setminus \{a\}$ is open.

2. Find an infinite non open subset of $\mathbb{Z}$. (5 pts.)

   Answer: The set of primes is not an open subset. Because otherwise, for some $a \neq 0$ and $b \in \mathbb{Z}$, the elements of $a\mathbb{Z}+b$ would all be primes. So b, $ab+b$ and $2ab+b$ would be primes, a contradiction.

3. Let $a,\, b \in \mathbb{Z}$. Is the map $f_{a,b} : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f_{a,b}(z) = az+b$ continuous? (Prove or disprove). (10 pts.)

   Answer: Translation by b is easily shown to be continuous. Let us consider the map $f(z) = az$. If $a = 0, 1, -1$ then clearly f is continuous. Assume $a \neq 0, \pm 1$ and that f is continuous. We may assume that $a > 1$ (why?) Choose a b which is not divisible by a. Then $f^{-1}(b\mathbb{Z})$ is open, hence contains a subset of the form $c\mathbb{Z}+d$. Therefore $a(c\mathbb{Z}+d) \subseteq b\mathbb{Z}$. Therefore $ac = \pm b$ and so a divides b, a contradiction. Hence f is not continuous unless $a = 0, \pm 1$.

4. Is the map $f_{a,b} : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f(z) = z^2$ continuous? (Prove or disprove). (5 pts.)

   Answer: No! Left as an exercise.

5. Is the topological space $(\mathbb{Z}, \tau)$ compact? (Prove or disprove).(15 pts.)

   Answer: First Proof: Note first the complement of open subsets of the form $a\mathbb{Z}+b$ are also open as they are unions of the form $a\mathbb{Z}+c$ for $c = 0, 1, \ldots, a-1$ and $c \not\equiv b \bmod a$. Now consider sets of the form $U_p = p\mathbb{Z} + (p-1)/2$ for p an odd prime. Then $\cap_p U_p = \emptyset$ because if $a \in \cap_p U_p$ then for some $x \in \mathbb{Z} \setminus \{-1\}$, $2a+1 = px+p$, so that a is divisible by all primes p and $a = 0$. But if $a = 0$ then $(p-1)/2$ is divisible by p, a contradiction. On the other hand no finite intersection of the U_p's can be emptyset as $(a\mathbb{Z}+b) \cap (c\mathbb{Z}+b) \neq \emptyset$ if a and b are prime to each other (why?) Hence $(U_p^c)_p$ is an open cover of $\mathbb{Z}$ that does not have a finite cover. Therefore $\mathbb{Z}$ is not compact.

   First Proof: Let p be a prime and $a = a_0 + a_1 p + a_2 p^2 + \ldots$ be a p-adic integer which is not in $\mathbb{Z}$. Let

   $$b_n = a_0 + a_1 p + a_2 p^2 + \ldots + a_{n-1} p^{n-1}.$$

   Then $\cap_n p^n \mathbb{Z} + b_n = \emptyset$ but no finite intersection is empty. We conclude as above.

III. Miscellaneous.

1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the squaring map. Suppose that the arrival set is endowed with the usual Euclidean topology. Find the smallest topology on the domain that makes f continuous. (5 pts.)

   Answer: The smallest such topology is the set

$$\{U \cap -U : U \text{ open in the usual topology of } \mathbb{R}\}.$$

2. Let τ be the topology on $\mathbb{R}$ generated by $\{[a, b) : a,\, b \in \mathbb{R}\}$. Compare this topology with the Euclidean topology. (3 pts.) Is this topology generated by a metric? (20 pts.)

   Answer: Any open subset of the Euclidean topology is open in this topology because $(a, b) = \cup_{n=1}^{\infty}[a + 1/n, b)$. But of course $[0, 1)$ is not open in the usual topology.

   Assume a metric generates the topology. Note that $[0, \infty)$ is open as it is the union of open sets of the form $[0, n)$ for $n \in \mathbb{N}$. Thus the sequence $(-1/n)_n$ cannot converge to 0. In fact for any $b \in \mathbb{R}$, no sequence can converge to b from the left. Thus for any $b \in \mathbb{R}$ there is an $\epsilon_b > 0$ such that $B(b, \epsilon_b) \subseteq [b, \infty)$. Let b_0 be any point of $\mathbb{R}$. Let $\epsilon_0 > 0$ be such that $B(b_0, \epsilon_0) \subseteq [b_0, \infty)$. Since $\{b_0\}$ is not open, there is $b_1 \in B(b_0, \epsilon_0) \setminus \{b_0\}$. Let $0 < \epsilon_1 < \epsilon_0/2$ be such that $B(b_1, \epsilon_1) \subseteq [b_1, \infty) \cap B(b_0, \epsilon_0)$. Inductively we can find $(b_n)_n$ and $(\epsilon_n)_n$ such that $B(b_n, \epsilon_n) \subseteq [b_n, \infty) \cap B(b_{n-1}, \epsilon_{n-1}) \setminus \{b_{n-1}\}$ and $\epsilon_n < \epsilon_0/2^n$. Then $(b_n)_n$ is a strictly increasing convergent sequence, a contradiction.

3. Show that the series $\sum_{i=0}^{n} x^n/n!$ converges for any $x \in \mathbb{R}$ (10 pts.) Show that the map $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\exp(x) = \sum_{i=0}^{n} x^n/n!$ is continuous. (10 pts.)

   Answer: For the first part show that the sequence of partial sums is Cauchy. The second part is easy as well, just write down.