MATH 221: Problem set # 4
Deadline date: 24.01.02

1. Prove that the function

$$f(x,y) = \frac{x-y}{x+y}$$

has no limit at $(0,0)$.

2. Find the limit

$$\lim_{(x,y)\to(0,a)} \frac{\sin xy}{x}.$$

3. Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } x^2+y^2>0, \\ 0, & \text{if } x^2+y^2=0. \end{cases}$$

is continuous at $(0,0)$.

4. Is the function $f : E \to \mathbf{R}$,

$$f(x,y) = \sin\frac{\pi}{1-x^2-y^2}$$

where $E = B((0,0),1)$. uniformly continuous on E?

5. Suppose that $A : \mathbf{R}^n \to \mathbf{R}^m$ is a linear mapping. Prove that for every point $\overline{x} \in \mathbf{R}^n$

$$A'(\overline{x}) = A.$$

6. For the following functions $f, g : \mathbf{R}^2 \to \mathbf{R}^2$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x^2+y^2 \end{pmatrix} \text{ and } g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2+y^2 \\ y \end{pmatrix}$$

calculate the Jacobians and check that

$$J_{f\circ g}(x,y) = J_f(g(x,y))J_g(x,y)$$

that is, the Chain Rule in the terms of Jacobians.

7. Is the function

$$f(x,y) = \sqrt[3]{xy}$$

differentiable at $(0,0)$?

8. Let

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } x^2+y^2>0, \\ 0, & \text{if } x^2+y^2=0. \end{cases}$$

Show that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

9. Find the first and second order partial derivatives of the following functions:

- $u = x^4 + y^4 - 4x^2y^2$;
- $u = xy + \dfrac{x}{y}$;
- $u = \dfrac{x}{y^2}$;
- $u = \dfrac{x}{\sqrt{x^2+y^2}}$;
- $u = x\sin(x+y)$;

- $u = \dfrac{\cos x^2}{y}$;
- $u = \tan \dfrac{x^2}{y}$;
- $u = x^y$;
- $u = \ln(x + y^2)$;
- $u = \arctan \dfrac{y}{x}$.
- $u = \arctan \dfrac{x+y}{1-xy}$;
- $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$;
- $u = \left(\dfrac{x}{y}\right)^z$;
- $u = x^{y/z}$;
- $u = x^{y^z}$;

10. For the function

$$u = \ln \sqrt{x^2 + y^2}$$

find

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

11. Find the first and second order partial derivatives for the following *compositions* (f is supposed to be 2-times differentiable):

- $u = f(x^2 + y^2 + z^2)$;
- $u = f\left(x, \dfrac{x}{y}\right)$;
- $u = f(x, xy, xyz)$.

12. Show that the function $z = x^n f\left(\dfrac{y}{x^2}\right)$, where f is an arbitrary differentiable function, satisfies the following (differential) equation

$$x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = nz.$$

13. Find points of local minima and maxima for the function

$$f(x, y) = xy \ln(x^2 + y^2).$$