The Taylor Formula

The idea of representing of a sufficiently differentiable function f in a small neighbourhood of a point a by series

$$\sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a) \tag{0.1}$$

or by a polynomial

$$\sum_{k=0}^{m} \frac{(x-a)^k}{k!} f^{(k)}(a) \tag{0.2}$$

came from English mathematical school of XVIII century.

Taylor (1715) considers the series in (0.1) as the 'limit case' of interpolation polynomials $f_m(x)$ of degree m which coincide with a function in the points

$$a, a + \Delta x, a + 2\Delta x, \ldots, a + (m-1)\Delta x$$

It is postulated then that the function

$$\lim_{\Delta x \to 0} f_m(x)$$

is equal to the polynomial in (0.2) and that

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a). \tag{0.3}$$

Maclaurin (1742) comes to the same idea determining coefficients of a polynomial

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_m(x-a)^m$$

which satisfies the conditions

$$f^{(k)}(a) = p^{(k)}(a)$$

for all $k = 1, \ldots, m$.

As it had been demonstrated later by Cauchy (1823) the formula (0.3) is not true in general. As the counterexample Cauchy considers the function

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases},$$

which is everywhere continuous. But one can prove that

$$f^{(k)}(0) = 0.$$

Thus, the Taylor series at $a = 0$ for f is $0 + 0 + \ldots + 0 + \ldots$ and the formula (0.3) does not hold.

However, the idea of representing of a sufficiently differential function in the form

$$f(x) = \sum_{k=0}^{m} \frac{(x-a)^k}{k!} f^{(k)} + R_m(x),$$

where the remainder $R_m(x)$ is 'small', is very useful. The problem is to choose an *appropriate form* of the remainder $R_m(x)$; there is a number of classical results in this direction. The first one was obtained in 1797 by Lagrange.

THEOREM 0.1 (Lagrange). *Let $f : [a, b] \to \mathbf{R}$ be a continous function which is $(m+1)$-times differentiable on (a, b). Then for every $x \in (a, b)$*

$$f(x) = \sum_{k=0}^{m} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{(x-a)^{k+1}}{(k+1)!} f^{(m+1)}(\xi),$$

where $\xi \in (a, x)$.

PROOF. The proof is based on the following theorem (Cauchy).

THEOREM 0.2. *Let $f.g : [a, b] \to \mathbf{R}$ be functions continuous on $[a, b]$ and differentiable on (a, b). If $g'(x) \neq 0$ for $a < x < b$, then $g(b) \neq g(a)$ and exists $\xi \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

PROOF. Recall that Rolle's Theorem states that for every function h which is continous on $[a, b]$ and differentiable on (a, b) the condition $h(a) = h(b)$ implies that there is a zero of the derivative $h'(x)$ on $(a, b) : h(\eta) = 0$ and $\eta \in (a, b)$.

Applying Rolle's theorem to g, we see that $g(a) \neq g(b)$. For the differentiable on (a, b) function

$$s(x) = f(x) - f(a) + (g(x) - g(a)) \frac{f(b) - f(a)}{g(b) - g(a)}$$

we have that $s(a) = s(b)$. Again by Rolle's theorem there is a zero of $s'(x)$ on (a, b) and the result follows. □

Consider the remainder

$$R_m(x) = f(x) - \sum_{k=0}^{m} \frac{(x-a)^k}{k!} f^{(k)}(a)$$

and compare it with the function

$$S_m(x) = \frac{(x-a)^{m+1}}{(m+1)!}$$

We have that

$$R(a) = 0, \quad R'(a) = 0, \ldots, R^{(m)}(a) = 0$$

and, similarly,

$$S_m^{(k)} = 0.$$

for every $k = 0, \dots, m$. Applying Theorem 0.2 repeatedly we get

$$\frac{R_m(x)}{S_m(x)} = \frac{R_m(x) - R_m(a)}{S_m(x) - S_m(a)} = \frac{R'_m(\xi_1)}{S'_m(\xi_1)} = \frac{R'_m(\xi_1) - R'_m(a)}{S'_m(\xi_1) - S'_m(a)}$$

$$= \frac{R''_m(\xi_2)}{S''_m(\xi_2)} = \frac{R''_m(\xi_2) - R''_m(a)}{S''_m(\xi_2) - S''_m(a)} = \dots = \frac{R_m^{(m+1)}(\xi_{m+1})}{S_m^{(m+1)}(\xi_{m+1})}, \tag{0.4}$$

where

$$a < \xi_1 < x, \quad a < \xi_2 < \xi_1, \dots, a < \xi_{m+1} < \xi_m.$$

Since $R^{(m+1)}(x) = f^{(m+1)}(x)$ and $S_m^{(m+1)}(x) = 1$, we obtain from (0.4) that

$$R_m(x) = S_m(x) f^{(m+1)}(\xi)$$

with $\xi = \xi_{m+1}$. This completes the proof of the theorem. □