ANALYSIS 1

Questions

1. Find the following limits:

(a) $\lim_{x \longrightarrow 3} \dfrac{x^3 - 8}{x - 2}$
(b) $\lim_{x \longrightarrow 0} \dfrac{1 - \sqrt{1 - x^2}}{x}$
(c) $\lim_{h \longrightarrow 0} \dfrac{\sqrt{x + h} - \sqrt{x}}{h}$

2.

(a) Prove that $\lim_{x \longrightarrow a} f(x) = \lim_{h \longrightarrow 0} f(x + h)$.
(b) Prove that $\lim_{x \longrightarrow 0} f(x) = \lim_{x \longrightarrow 0} f(x^3)$.

3.

(a) Let E be a metric space. Prove that if $f : E \longrightarrow \mathbb{R}$ is continuous at a, then $|f|$ is also continuous at a.
(b) Show also that if f is a polynomial function from $\mathbb{R}$ to $\mathbb{R}$ then there is a $y \in \mathbb{R}$ such that $|f(y)| \leq |f(x)|$ for all $x \in \mathbb{R}$.

4. Let E be the set of all continuous functions $f : [0,1] \longrightarrow \mathbb{R}$. For $f \in E$ define $\|f\| = \sup_{x \in [0,1]} |f(x)|$. Show that for any number c and $f \in E$ we have $\|cf\| = |c|\|f\|$. Show also that $\|f\| = 0$ iff $f = 0$ and that $\|f + g\| \leq \|f\| + \|g\|$ for each $f, g \in E$. Find examples to show that $\|f + g\| \neq \|f\| + \|g\|$.

5. Let X be a metric space and let (x_n) be sequence in X converging to $x \in X$. Show that $K := \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact.

6. Find the sum of the first $n + 1$ terms of the series

$$1 + 2x + 3x^2 + 4x^3 + ... + (n + 1)x^n + ...$$

and show that if $|x| < 1$ the series converges to $\dfrac{1}{(1 - x)^2}$.

7. Show that $(x_n)_{n \geq 1}$ is a convergent sequence of real numbers with limit a then so is

$$\left(\frac{x_1 + x_2 + x_3 + ... + x_n}{n} \right)_{n \geq 1}.$$

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