MATH 283: ANALYSIS 3

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Midterm Questions

1. (a) Let $\mathbf{a}$ be a fixed vector, T be fixed linear transformation of $\mathbb{R}^n$. Define $F(\mathbf{x}) = (\mathbf{a} \cdot \mathbf{x})T(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^n$. Show that

$$F'(\mathbf{x})(\mathbf{y}) = (\mathbf{a} \cdot \mathbf{y})T(\mathbf{x}) + (\mathbf{a} \cdot \mathbf{x})T(\mathbf{y})$$

for all $x, y \in \mathbb{R}^n$.
(b) Let $a \in \mathbb{R}$ and define $f(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{x})^a$ for each $\mathbf{x} \in \mathbb{R}$. Show that $\nabla f(\mathbf{x}) = 2a(\mathbf{x} \cdot \mathbf{x})^{a-1}\mathbf{x}$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ find $(\nabla f)'(\mathbf{x})(\mathbf{y})$.
2. Is every vector field on $\mathbb{R}^n$ the gradient of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$? Prove or disprove.
3. (a) Suppose $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function such that

$$\frac{\partial f}{\partial y}(a, b) = 0 \qquad \forall (a, b) \in \mathbb{R}^2.$$

Show that f is independent of the second variable.
(b) For $x, y \in \mathbb{R}$ we define

$$f(x, y) = x^{x^y} + \log(x)[\arctan(\arctan(\sin(\cos(xy) - \log(x + y))))].$$

For $b \in \mathbb{R}$ compute $\dfrac{\partial f}{\partial y}(1, b)$.
4. (a) Expand $f(x, y, z) = xyz$ by Taylor's formula about $x_0 = (1, -1, 0)$, with $q = 4$.
(b) Let φ be a function of class $C^{(q)}$ in an open neighborhood of 0, $a, b \in \mathbb{R}$. Define $f(x, y) = \varphi(ax + by)$ for each $(x, y) \in \mathbb{R}^2$. Show that the Taylor formula about $(0, 0)$ becomes

$$f(x, y) = \sum_{k=0}^{q-1} \frac{\varphi^{(k)}(0)}{m!} \sum_{j=0}^{k} \binom{m}{j} (ax)^j (by)^{k-j} + R_q(x, y).$$

5. Suppose $B \subseteq \mathbb{R}$ is open, $a < b$ are real numbers, $f(x, t)$ and $\dfrac{\partial f}{\partial t}(x, t)$ are continuous functions on $[a, b] \times B$. Let p, q be two continuous functions on B with values in $[a, b]$. Put

$$G(x, t) = \int_a^x f(s, t)ds.$$

We know from fundamental theorem of calculus that $\frac{\partial G}{\partial x} = f$. Compute the derivative of $G[q(t), t] - G[p(t).t]$ and deduce that

$$\frac{d}{dt}\left[\int_{p(t)}^{q(t)} f(x, t)dx\right] = f(q(t), t)q'(t) - f(p(t), t)p'(t) + \int_{p(t)}^{q(t)} \frac{\partial f}{\partial t}(x, t)dx.$$

6. For $t > 1$ define $\phi(t) = \int_{\log t}^{t} x^{-1} \exp(x^2 t^2)dx$. Use question **5** to find $\phi'(t)$.
7. Let $\mathbf{a}$ be a fixed vector in $\mathbb{R}^n$, φ be a function of class $C^{(2)}$ on $\mathbb{R}$ and define $f(\mathbf{x}) = \varphi(\mathbf{a} \cdot \mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^n$. Show that every critical point of f is degenerate.
8. Let $f(x, y) = x \cos y$. Find the differential df, the Jacobi matrix, and the Hesse matrix of f. Find also the critical points of f and determine whether they are extremum points.
9. Define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ as follows:

$$f(r, \theta, \phi) := (r \cos\theta \sin\phi, r \sin\theta \sin\phi, r \cos\phi).$$

Find the Jacobi matrix of f at the point (a, b, c).
10. Let p be a real number. If $f(t\mathbf{x}) = t^p f(\mathbf{x})$ for all $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$ and $t > 0$, then show that

$$df(\mathbf{x}) \cdot \mathbf{x} = pf(\mathbf{x})$$

for every $\mathbf{x} \neq \mathbf{0}$. Prove also that the converse is true.