Math 111 Problems

(Ali Nesin)

1. Define

   (a) $x \subseteq y$ (x is a subset of y),

   (b) $x \subset y$ (x is a proper subset of y).

2. Show that $\emptyset$ is a subset of every set.

3. Show that every element of $\emptyset$ is equal to$\sqrt{2}$.

4. Show that every element of $\emptyset$ is equal to empty set.

5. Let A be a set. Show that $\{a \in A : a \notin a\} \notin A$.

6. Show that if $X \subseteq \emptyset$ then $X = \emptyset$.

7. We know that if X is nonempty, then the intersection of all elements of X is a set. What happens if $X = \emptyset$?

8. Consider $\forall y(\forall z(z \notin x \wedge z \notin y) \Longrightarrow x = y)$. For what x's does this hold?

9. Call $X \subseteq \mathbb{R}^2$ open if and only if $\forall (x, y) \in X \exists \varepsilon > 0 B(x, \varepsilon) \subseteq X$.Show that intersection of finitely many open subsets of $\mathbb{R}^2$ is open.

10. Call a set *P-set* if it has property P. Assume that the intersection of *P-sets* is a *P-set*. For $A \subseteq X$ define

    $\overline{A} = \bigcap_{A \subseteq Y \subseteq X} Y$ where all$Y$'s are *P-sets*. Compare $\overline{A \cap B}$ and $\overline{A} \cap \overline{B}$.

11. Call a subset of $X \subseteq \mathbb{R}$ closed if for any $x \in \mathbb{R} \backslash X$ there is an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap X = \emptyset$.

    (a) Show that $[0, 1]$ is closed.

    (b) Show that $[0, 1)$ is not closed.

    (c) Show that the intersection of closed sets is closed.

    (d) Find the closure of $\{\frac{1}{n} : n = 1, 2, 3, ...\}$.

    (e) Find the closure of $\mathbb{N}$ and $[0, 1]$.

    (f) Find the closure of $(0, 1)$.

    (g) Find the closure of $\mathbb{Q}$.

    (h) Find the closure of $\{\frac{a}{2^n} : a, n \in \mathbb{Z}\}$.

12. Show that the union of finitely many closed sets is closed. But the union of closed sets is not always closed.

13. Show that X is closed if $R \backslash X$ is open.

14. Let us call a subset X of $\omega \times \omega \times \omega$ nice if

    i)$(x, 0, x) \in X$

    ii)$(x, y, z) \in X \Longrightarrow (x, Sy, Sz) \in X \quad$ for all $x, y, z \in \omega$.

    Show that

    (a) $\omega \times \omega \times \omega$ is a nice set,
    (b) intersection of all nice sets A is the smallest nice set,
    (c) A is a function from $\omega \times \omega$ to ω.

15. Show that the multiplication is a function from $\omega \times \omega$ to ω.

16. Let A be a set and n be an integer. Define $A^n = \{f : n \longrightarrow A\}$.

    (a) Show that $B = \bigcup_{n \in \mathbf{N}} A^n$ is a set.
    (b) Show that there is a function $\sum$ from B to ω such that $\sum(b) = \sum_{i=1}^{n} b_i$.

17. Show that α is a limit ordinal iff $\bigcup \alpha = \alpha$.

18. Show that ${}^A 2 \sim \wp(A)$.

19. Show that ${}^C({}^B A) \sim^{C \times B} A$.

20. Show that ${}^C(A \times B) \sim ({}^C A) \times ({}^C B)$.

21. Show that if $(a_n)_n$ is Cauchy and $\lim a_n \neq 0$ then $(a_n)_n$ is away from zero after a while.

22. Show that if $(q_n)_n$ and $(q'_n)_n$ are Cauchy then $(q_n \pm q'_n)_n$ and $(q_n . q'_n)_n$ are also Cauchy.

23. Show that if $(q_n)_n$ and $(q'_n)_n$ are Cauchy, $\lim q'_n \neq 0$, then $(q_n / q'_n)_n$, is also Cauchy.

24. Show that $(1 + \frac{1}{n})^n$ is a Cauchy sequence.

25. Show that every Cauchy sequence is bounded.

26. Show that every convergent sequence is Cauchy.

27. Show that every Cauchy sequence is convergent.

28. Let $(q_n)_n$ be Cauchy. Assume that a subsequence of $(q_n)_n$ converges to l. Show that $(q_n)_n$ converges to l.

29. Let C be the ring of all Cauchy sequences over rational numbers. Show that the ring of all sequences converging to zero is a maximal ideal of C.

30. Show that a monotone sequence can have at most one cluster point.

31. If $(x_n)_n$ is bounded and has only one cluster point, then $(x_n)_n$ converges to that cluster point.

32. Show that for any $\varepsilon \in \mathbb{Q}^{>0}$ there is an $n \in \mathbb{N}$ such that $\frac{1}{10^{n-1}} < \varepsilon$.

33. Can you find a sequence with infinitely many cluster points?

34. Let $\sum a_n$ be a convergent series. Show that $\lim a_n = 0$.

35. Let $0 \leq r < 1$. Assume that $a_{n+1} \leq a_n$. Show that $\sum a_n$ converges.

36. Suppose that $0 \leq a_n$ and $\lim \frac{a_{n+1}}{a_n}$ exists and < 1. Show that $\sum a_n$ converges.

37. Show that for any $x \in R$, $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ is Cauchy. Define $f(x) = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$. Show that f is differentiable. Find its derivative.

38. Let $x_n \in \mathbb{Q}^{>0}$. Assume that $(x_n^2)_n$ is Cauchy. Show that $(x_n)_n$ is Cauchy.

39. Let $(x_n)_n$ be a sequence such that for some $r < 1$, $|x_{n+1} - x_n| < |x_n - x_{n-1}|$. Show that $(x_n)_n$ converges. Show that we cannot take $r = 1$.

40. Find a function from $\mathbb{R}$ to $\mathbb{R}$ such that it is continuous at one and only one point.

41. Find a function such that f is not continuous on any point on $\mathbb{R}$ but fof is continuous everywhere.

42. Let f be a function such that $f(x+y) = f(x) + f(y)$. Prove that if f is continuous at 0, then f is continuous on $\mathbb{R}$.

43. Let f be a continuous function from (a, b) to (a, b) which is injective. Show that f is monotone.

44. Let f and g be continuous functions such that $f(x) = g(x)$ for all $x \in (a, b) \cap Q$. Then $f(x) = g(x)$ for all $x \in (a, b)$.

45. Prove that $e^x - 3x - \sin x = 0$ has a solution in $(0, 1)$.

46. On ω define $a\beta b$ iff $a \geq b$. Show that β is not a well order.

47. On $\omega \times \omega$ define $(a, b) \leq (x, y)$ iff $a < x$ or ($a = x$ and $b < y$). Show that this is a well order.

48. Show that there is no strictly decreasing function from ω into a well ordered set.

49. On $\omega \times \omega$ define $(a, b) < (x, y)$ iff $(2a+1).2^y < (2x+1).2^b$. Show that this is a total order which is not a well order.

50. Let X be a set, let $A \subset \wp(X)$, such that A has finite intersection property.

    (a) Show that the set of filters containing A is closed under arbitrary intersection.

    (b) Conclude that "the filter generated by A" exists. Denote it by $\langle A \rangle$.

    (c) Show that $\langle A \rangle = \{a \subset X : \exists a_1, ..., a_n \in A \text{ such that } \bigcap\limits_{i=1}^{n} a_i \subset a\}$.

    (d) Show that there is an ultrafilter containing A.