MATH 111

Homework 10

Notation: Let f be a function from a set X to itself. Then for any natural number n, $f^{(n)}$ denotes the composition of f with itself n times. The symbol X^X denotes the set of all functions from X to X. The symbol Id_X stands for the identity map on X.

1. Let X be a set and $f : X \to X$ be a function. In this exercise we will show that $\{f^{(n)} : n \in N\}$ is a set and $\varphi : N \to X^X$ defined by $\varphi(n) = f^{(n)}$ is a function.

   (a) Call a subset Y of $N \times X^X$ an **f-set** if

      i. $(0, Id_X) \in Y$

      ii. for any $n \in N$, $(n, g) \in Y \Longrightarrow (n+1, f \circ g) \in Y$

      Give three examples of f-sets.

   (b) Show that the collection of all f-sets is a set .

   (c) Show that the intersection of all f-sets is an f-set, call it G.

   (d) Show that for any natural number n, $(n, g) \in G \Longrightarrow g = f^{(n)}$. (Hint: proceed by induction on n and use the fact that G is the smallest f-set.)

   (e) Conclude that φ is a function and $\{f^{(n)} : n \in N\}$ is a set.

2. Using the first exercise, show that $\{f^{(n)}(X) : n \in N\}$ is a set.

3. Let $x_0 \in X$ and $f : X \to X$ be a function. Define $g(0) = x_0$ and $g(n+1) = f(g(n))$. Show that g is a function and $\{x_0, f(x_0), f^{(2)}(x_0), ...\}$ is a set.