MATH 111
Homework 12

Notations: AC denotes the Axiom of Choice, ZL denotes Zorn's Lemma, WOP denotes the Well-Ordering Principle.

1. In this exercise, we will prove that AC implies WOP without using ZL. The proof is due to *Zermelo*.

   Let A be a nonempty set. Let $g : P(A)\backslash\{\emptyset\} \longrightarrow A$ be a *choice function* . Call a subset T of $P(A)$ a **tower** if

   (a) $B \in T$ and $B \neq A$ implies $B \cup \{g(A\backslash B)\} \in T$ for all $B \subseteq A$,

   (b) the union of any chain in T is an element of T (thus $\emptyset \in T$).

2. Show that there is a unique smallest tower, call it T_0.

3. Call an element C of T_0 comparable if either $B \subseteq C$ or $C \subseteq B$ for all $B \in T_0$. Show that the set of comparable elements of T_0 is a tower. (Hint: remember the proof of AC implies ZL) Deduce that T_0 is a chain.

4. Show that $\cup T_0 = A$ (assume not and use the second condition of being a tower)

5. Show that for any element $x \in A$, there is a unique largest $C \in T_0$ such that $x \notin C$ and such that for this C we have $x = g(A\backslash B)$ (try the union of all the elements of T_0 not containing x)

6. Deduce that A is well-ordered by $<$, where

$$x < y \Longleftrightarrow (\exists B \in T_0)\ ((x \in B) \wedge (y \notin B))$$