1 Solutions for HW4

1. Let $f : X \longrightarrow Y$ be a function. Define a relation $\sim$ on X as follows:

$$x \sim y \quad \Longleftrightarrow \quad f(x) = f(y)$$

   (a) Let $x \in X$. Since $f(x) = f(x)$, $\sim$ is reflexive.
   Let $x \sim y$. Then $f(x) = f(y)$. Thus $y \sim x$ too. Hence $\sim$ is symmetric.
   Let $x \sim y$ and $y \sim z$. Then $f(x) = f(y) = f(z)$. Thus $x \sim z$. Hence $\sim$ is transitive.
   Thus $\sim$ is an equivalence relation on X.

   (b) We have to show that F is well-defined; that is we have to show that

$$\text{if} \quad [x] = [y] \quad \text{then} \quad F([x]) = F([y])$$

   Let $x, y \in X$ such that $[x] = [y]$. Then $x \sim y$. Thus $f(x) = f(y)$. This means that $F([x]) = F([y])$.

   (c) Let us show that F is 1-1. Let $F([x]) = F([y])$. Then $f(x) = f(y)$. This means $x \sim y$. Hence $x \in [y]$, $y \in [x]$. Thus $[x] = [y]$, as desired.
   By def. , $im\ F = \{y \in Y \mid y = F([x]),$ for some $x \in X\} = \{y \in Y \mid y = f(x)$ for some $x \in X\ \} = im\ f$.
   Hence f ans F have the same image sets.

2. Let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be two functions. Define $h : X \longrightarrow Z$ to be the composition $g \circ f$.

   (a) Assume that f, g are 1-1. We will show that h is also 1-1. Let $x, y \in X$ be such that $h(x) = h(y)$. Then $g(f(x)) = g(f(y))$. Since g is 1-1, we have $f(x) = f(y)$. From the injectivity of f, it follows that $x = y$.

   (b) Assume that f, g are onto. Let $z \in Z$. Surjectivity of g implies that there is $y \in Y$ such that $g(y) = z$. Now surjectivity of f implies that there is $x \in X$ such that $f(x) = y$. Hence $h(x) = g(f(x)) = z$.

   (c) Assume that f and g are bijections. From (a),(b) above, it follows that h is a bijection too.

   (d) We have to show that $h^{-1}(z) = f^{-1}(g^{-1}(z))$ for all $z \in Z$. Take any z in Z. Let $x \in X$ be the image of z under h^{-1}, that is $h(x) = z$. Then applying $f^{-1} \circ g^{-1}$ to z, we get $f^{-1}(g^{-1}(z)) = f^{-1}(g^{-1}(h(x))) = f^{-1}(g^{-1}(g(f(x)))) = x$.

3. Let $F : X \longrightarrow Y$, $G : Y \longrightarrow Z$ be two functions. Define $H : X \longrightarrow Z$ to be the composition $G \circ F$.

   (a) Assume that H is 1-1. Let us show that F is 1-1 as well. Assume F is not 1-1. Then there is $x, y \in X$ such that $F(x) \neq F(y)$. Applying G now we get $H(x) = G(F(x)) = G(F(y)) = H(y)$. A contradiction.

(b) Assume that H is onto. Let us show that G is onto. Assume G is not onto. Then there exists some $z \in Z$ such that there is no $y \in Y$ with $G(y) = z$. Since H is onto, thereis some $x \in X$ be such that $H(x) = z$. Thus G sends $F(x)$ to z. A contradiction.

(c) Assume that H is a bijection. Then by (a),(b) we have F 1-1 and G onto. But F can be not onto and G can be not 1-1 as well. Consider the following example. Take $X = \{1\}$, $Y = \{2,3\}, Z = \{4\}$. Let $F = ((X,Y), \{(1,2)\})$ and $G = \{(Y,Z), \{(2,4),(3,4)\}\}$ (the functions are written in this way just to remind you the definition of function!). Then clearly, $H = G \circ F$ is bijective while F is not onto and g is not 1-1.

4. May be this question should have been asked as follows:

Find out the error in the following fake proof of the proposition "a transitive and symmetric relation R on a set X is reflexive"

Proof: Let $x \in X$. Since R is symmetric, $(x,y) \in R$ implies $(y,x) \in R$. Now applying transitivity to (x,y) and (y,x) we get $(x,x) \in X$ as desired.

Consider the following set and relation: let $X = \{1,2\}$ and $R = \{(2,2)\}$. Clearly R is transitive and symmetric but not reflexive.