Naive Set Theory (Math 111)
First Midterm

Fall 2002
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1. a) Given a set X, define $\cup X$ as follows:

$$y \in \cup X \text{ if and only if there is an } x \in X \text{ such that } y \in x.$$

   Show that $\cup\emptyset = \emptyset$. (3 pts.)

   b) Given a set X, define $\cap_1 X$ and $\cap_2 X$ as follows:

$$\begin{array}{lll} y \in \cap_1 X & \text{if and only if} & y \in x \text{ for all } x \in X \\ y \in \cap_2 X & \text{if and only if} & y \in \cup X \text{ and } y \in x \text{ for all } x \in X \end{array}$$

   Is $\cap_1 X = \cap_2 X$ for all X? Which definition do you prefer for $\cap X$ and why? (5 pts.)

2. Find a set X such that $X \cap \wp(X) \neq \emptyset$. (3 pts.)

3. Let $(A_i)_{i \in I}$ be a family of sets.

   a) Show that $\bigcap_{i\in I} \wp(A_i) = \wp(\bigcap_{i\in I} A_i)$. (3 pts.)

   b) What is the relationship between $\bigcup_{i\in I} \wp(A_i)$ and $\wp(\bigcup_{i\in I} A_i)$? (4 pts.)

4. What is the set $(X \times Y) \cap (Y \times X)$? (3 pts.)

5. Let α be a set such that $x \subseteq \alpha$ for all $x \in \alpha$. Show that $\alpha \cup \{\alpha\}$ has the same property. Give four examples of such sets. (4 pts.)

6. Let Γ be a graph such that for any vertices α, α_1, β, β_1, if $\alpha \neq \alpha_1$ and $\beta \neq \beta_1$, then there is a $\phi \in \mathrm{Aut}(\Gamma)$ such that $\phi(\alpha) = \beta$ and $\phi(\alpha_1) = \beta_1$. What can you say about Γ? (5 pts.)

7. Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be such that $\phi(x+y) = \phi(x)+\phi(y)$ and $\phi(x^2) = \phi(x)^2$ for all x, $y \in \mathbb{R}$. Show that $\phi(xy) = \phi(x)\phi(y)$ for all x, $y \in \mathbb{R}$ and $\phi(q) = q$ for all $q \in \mathbb{Q}$. (10 pts.)

8. Given a set X, define $\wp^n(X)$ as follows by induction on n: $\wp^0(X) = X$ and $\wp^{n+1}(X) = \wp(\wp^n(X))$.

   a) Is there a natural number n such that for any set X, $\{\{\emptyset\}, \{\{X\}\}\} \in \wp^n(X)$? (8 pts.)

   b) Show that $\wp(\wp^n(X)) = \wp^n(\wp(X))$ for all sets X and all natural numbers n. (8 pts.)

   c) Show that $\wp^n(\wp^m(X)) = \wp^m(\wp^n(X))$ for all sets X and all natural numbers n and m. (8 pts.)

9. Define a partial order $\prec$ on $\mathbb{N} \setminus \{0, 1\}$ by $x \prec y$ if and only if $x^2 | y$. Describe all the automorphisms of this poset. (5 pts.)

10. Let X be a set. Let Γ be the set of subsets of X with two elements. On Γ define the relation $\alpha R \beta$ if and only if $\alpha \cap \beta = \emptyset$. Then Γ becomes a graph with this relation.

    a) Calculate $\mathrm{Aut}(\Gamma)$ when $|X| = 4$. (3 pts.)

    b) Draw the graph Γ when $X = \{1, 2, 3, 4, 5\}$. (3 pts.)

    c) Show that $\mathrm{Sym}(5)$ imbeds in $\mathrm{Aut}(\Gamma)$ naturally. (You have to show that each element σ of $\mathrm{Sym}(5)$ gives rise to an automorphism $\tilde{\sigma}$ of Γ in such a way that the map $\sigma \mapsto \tilde{\sigma}$ is an injection from $\mathrm{Sym}(5)$ into $\mathrm{Aut}(\Gamma)$ and that $\widetilde{\sigma_1 \circ \sigma_2} = \widetilde{\sigma_1} \circ \widetilde{\sigma_2}$). (8 pts.)

    d) Show that $\mathrm{Aut}(\Gamma) \simeq \mathrm{Sym}(5)$. (12 pts.)