Set Theory (Math 111)
Final

Ali Nesin

January 11, 2004

You may assume that you know all the basic arithmetic properties of $(\mathbb{Z}, +, \times, 0, 1)$ and $(\mathbb{N}, +, \times, 0, 1)$.

1. Let $X = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Define the relation $\equiv$ on X by
$$(x, y) \equiv (z, t) \Leftrightarrow xt = yz$$
for every (x, y), $(z, t) \in X$.

   a) Show that this is an equivalence relation on X.

   b) Find the equivalence classes of $(0, 1)$ and of $(3, 3)$.

   c) Show that if $(x, y) \equiv (x', y')$ and $(z, t) \equiv (z', t')$ then $(xt + yz, yt) \equiv (x't' + y'z', y't')$.

   d) Show that if $(x, y) \equiv (x', y')$ and $(z, t) \equiv (z', t')$ then $(xz, yt) \equiv (x'z', y't')$.

2. Find a graph which has only three automorphisms.

3. Let a and b be two integers. We say that d is the **greatest common divisor** of a and b if d is the largest natural number that divides both a and b. We let $d = \gcd(a, b)$. Show that for any a, $b \in \mathbb{Z}$, $\gcd(a, b)$ exists and that there are x, $y \in \mathbb{Z}$ such that $ax + by = \gcd(a, b)$.

4. Let a and b be two integers. We say that m is the **least common multiple** of a and b if m is the least natural number that is divisible by both a and b. We let $m = \mathrm{lcm}(a, b)$. Show that for any a, $b \in \mathbb{Z}$, $\mathrm{lcm}(a, b)$ exists and that $ab = \pm \gcd(a, b)\,\mathrm{lcm}(a, b)$.

5. Find formulas for the sums
$$1^2 + 2^2 + \ldots + n^2$$
and
$$1^3 + 2^3 + \ldots + n^3,$$
and prove your result.

6. Recall that a natural number $p \neq 0,\ 1$ is called **prime** if whenever p divides a product xy of two natural numbers x and y then p divides either x or y. A natural number $p \neq 0$, 1 is called **irreducible** if whenever $p = xy$ for two natural numbers x and y then either x or y is 1. Show that a natural number is prime if and only if it is irreducible.